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A NOTE ON MODIFIED DEGENERATE GAMMA RANDOM VARIABLES

TAEKYUN KIM, DAE SAN KIM, WONJOO KIM, AND JONGKYUM KWON

ABSTRACT. Let X and Y be independent random variables which are respectively the modified gamma random variable with parameters α_1 and β, and that with parameters α_2 and β. The aim of this note is to find the joint probability density function of $U = X+Y$ and $V = \frac{X}{X+Y}$, which gives a partial solution to an open problem.

1. INTRODUCTION

For any $\lambda \in \mathbb{R}$, the degenerate exponentials are defined by

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \quad (\text{see } [3,6-14]), \quad e_\lambda^1(t) = e_\lambda(t), \tag{1}$$

where

$$(x)_{0,\lambda} = 1,\ (x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda),\ (n \geq 1), \quad (\text{see } [1-7,9]). \tag{2}$$

It is known that the λ-binomial coefficients are given by

$$\binom{x}{n}_\lambda = \frac{(x)_{n,\lambda}}{n!} = \frac{x(x-\lambda)\cdots(x-(n-1)\lambda}{n!},\ (n \geq 1), \quad \binom{x}{0}_\lambda = 1, \quad (\text{see } [9,12,14]).$$

The degenerate gamma functions are defined by

$$\Gamma_\lambda^*(s) = \int_0^\infty e_\lambda^{-1}(t) t^{s-1} dt, \quad (\lambda \in (0,1)), \quad (\text{see } [6-8,17]). \tag{3}$$

Note that $\lim_{\lambda \to 0} \Gamma_\lambda^*(s) = \Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt \quad (\mathrm{Re}(s) > 0))$.

We recall that X is a continuous random variable if there exists a nonnegative function f, defined for all $x \in (-\infty,\infty)$, having the property that, for any set B of real numbers.

$$P\{x \in B\} = \int_B f(x)dx, \quad (\text{see } [3,11,15]).$$

The moments of X are given by

$$E[X^n] = \int_{-\infty}^{\infty} x^n f(x)dx, \quad (n \geq 0), \quad (\text{see } [11,15]). \tag{4}$$

In particular, for $n = 1$, $E[X]$ is the mean of X. Let $g(x)$ be a real valued function. Then the expectation of $g(X)$ is defined by

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx, \quad (\text{see } [11,15]). \tag{5}$$

The random variables X and Y are jointly continuous if there exists a nonnegative function $f_{X,Y}(x,y)$, defined for all $x,y \in \mathbb{R}$, having the probability that, for all sets A and B of real numbers

$$P\{X \in A,\ Y \in B\} = \int_B \int_A f_{X,Y}(x,y)dxdy = \int_{A\times B} f_{X,Y}(x,y)dxdy, \quad (\text{see } [3,11,15]). \tag{6}$$

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The function $f_{X,Y}(x,y)$ is called the joint probability density function of X and Y. The random variables X and Y are said to be independent if $E[XY] = E[X]E[Y]$ (see (4)). In general, for independent random variables X and Y, $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$ (see (5)), where g,h are real valued functions.

For $\lambda \in (0,1)$, a random variable X is said to have the degenerate gamma distribution with parameters α and β, ($\frac{1}{\lambda} > \alpha > 0,\ \beta > 0$), if its probability density function has the form (see (1), (2), (3))

$$f_X^*(x) = \begin{cases} \frac{1}{\Gamma_\lambda^*(\alpha)}\beta e_\lambda^{-1}(\beta x)(\beta x)^{\alpha-1}, & \text{if } x > 0, \\ 0, & \text{otherwise}, \end{cases} \quad \text{(see [11])}. \tag{7}$$

Then, from (3) and (7), we can show that $\int_{-\infty}^{\infty} f_X^*(x)dx = 1$. In this case, we say that X is the degenerate gamma random variable with parameters α and β, for which we denote by $X \sim \Gamma_\lambda^*(\alpha,\beta)$.

We recall the following open problem about the degenerate gamma random variables from [11].

Open Problem: Let $X \sim \Gamma_\lambda^*(\alpha_1,\beta)$, $Y \sim \Gamma_\lambda^*(\alpha_2,\beta)$, and let X and Y be independent with the joint probability density function $f_{X,Y}(x,y)$. Suppose that $U = X+Y$, $V = \frac{X}{X+Y}$. Find the joint probability density function of U and V.

We consider the modified degenerate gamma function $\Gamma_\lambda(s)$ (see (10)) and the modified degenerate gamma random variable with parameters α and β, $X \sim \Gamma_\lambda(\alpha,\beta)$ (see (11)), in place of the degenerate gamma function $\Gamma_\lambda^*(s)$ and the degenerate gamma random variable with parameters α and β, $X \sim \Gamma_\lambda^*(\alpha,\beta)$, and study their properties. The aim of this note is to give the answer to the following "Modified Open Problem" which gives a partial solution to the "Open Problem" in the above. We derive the solution from Theorem 2.2 which follows from the change of variables formula in classical analysis.

Modified Open Problem: Let $X \sim \Gamma_\lambda(\alpha_1,\beta)$, $Y \sim \Gamma_\lambda(\alpha_2,\beta)$, and let X and Y be independent with the joint probability density function $f_{X,Y}(x,y)$. Suppose that $U = X+Y$, $V = \frac{X}{X+Y}$. Find the joint probability density function of U and V.

2. Modified degenerate gamma random variables

To show Theorem 2.2, we need the following change of variables formula for integration (see [16]).

Lemma 2.1. *Suppose that T is a one-to-one C^1-mapping of an open set $E \subset \mathbb{R}^k$ into $\mathbb{R}^k$ such that the Jacobian determinant $J_T(x) \neq 0$, for all $x \in E$. If f is a continuous function on $\mathbb{R}^k$ whose support is compact and lies in $T(E)$, then*

$$\int_{\mathbb{R}^k} f(y)dy = \int_{\mathbb{R}^k} f(T(x))|J_T|(x)dx.$$

The next theorem is the key to our partial solution to the *Open Problem*.

Theorem 2.2. *Let X_1 and X_2 be jointly continuous random variables with the joint probability density function $f_{X_1,X_2}(x_1,x_2)$, and let $Y_1 = g_1(X_1,X_2)$, $Y_2 = g_2(X_1,X_2)$, where $g_1(x_1,x_2)$, $g_2(x_1,x_2)$ are real valued functions. Assume that the equations $y_1 = g_1(x_1,x_2)$, $y_2 = g_2(x_1,x_2)$ are uniquely solvable for x_1,x_2: $x_1 = h_1(y_1,y_2)$, $x_2 = h_2(y_1,y_2)$, with h_1 and h_2 continuously differentiable. Then the random variables Y_1 and Y_2 are jointly continuous with the joint probability density function given by*

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(h_1(y_1,y_2),h_2(y_1,y_2))\left|\frac{\partial(h_1,h_2)}{\partial(y_1,y_2)}\right|, \tag{8}$$

where $J_h = \frac{\partial(h_1,h_2)}{\partial(y_1,y_2)} = \det\begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{bmatrix}$ *is the Jacobian determinant.*

Proof. First, we note that $g=(g_1,g_2),\ h=(h_1,h_2)$ are maps from $\mathbb{R}^2$ onto itself, and $g\circ h = h\circ g = id$. Then, from (6), (8) and Lemma 2.1, we obtain

$$\begin{aligned}\int_{A\times B} f_{Y_1,Y_2}(y_1,y_2)dy_1dy_2 &= \int_{A\times B} f_{X_1,X_2}(h_1(y_1,y_2),h_2(y_1,y_2))\left|\frac{\partial(h_1,h_2)}{\partial(y_1,y_2)}\right| dy_1dy_2 \\ &= \int_{h(A\times B)} f_{X_1,X_2}(x_1,x_2)dx_1dx_2 \\ &= P\{(X_1,X_2)\in h(A\times B)\} \\ &= P\{g(X_1,X_2)\in A\times B\} \\ &= P\{(Y_1,Y_2)\in A\times B\} \\ &= P\{Y_1\in A,\ Y_2\in B\},\end{aligned}$$

which shows what we wanted. □

We note here that

$$\frac{\partial(h_1,h_2)}{\partial(y_1,y_2)}\frac{\partial(g_1,g_2)}{\partial(x_1,x_2)} = \det\begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{bmatrix}\det\begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{bmatrix} = 1. \tag{9}$$

The modified degenerate gamma function is defined by (see [13,14])

$$\Gamma_\lambda(s) = \int_0^\infty e_\lambda^{-t}(1)t^{s-1}dt, \quad (\mathrm{Re}(s)>0). \tag{10}$$

Note that

$$\Gamma_\lambda(s+1) = \frac{\lambda}{\log(1+\lambda)}s\Gamma_\lambda(s), \quad \lim_{\lambda\to 0}\Gamma_\lambda(s) = \Gamma(s).$$

For $\lambda\in(0,1)$, a random variable X is said to have the modified degenerate gamma distribution with parameters α an β, $(\frac{1}{\lambda}>\alpha>0,\ \beta>0)$, if its probability density function has the form

$$f_X(x) = \begin{cases} \frac{1}{\Gamma_\lambda(\alpha)}\beta e_\lambda^{-\beta x}(1)(\beta x)^{\alpha-1}, & \text{if } x>0, \\ 0, & \text{otherwise.}\end{cases} \tag{11}$$

In this case, we shall say that X is the modified degenerate gamma random variable with parameters α and β, for which we write as $X\sim\Gamma_\lambda(\alpha,\beta)$.
From (10) and (11), we note that

$$\begin{aligned}P\{X\in(-\infty,\infty)\} &= \int_{-\infty}^\infty f_X(x)dx = \frac{\beta}{\Gamma_\lambda(\alpha)}\int_0^\infty (\beta x)^{\alpha-1}e_\lambda^{-\beta x}(1)dx \\ &= \frac{1}{\Gamma_\lambda(\alpha)}\int_0^\infty y^{\alpha-1}e_\lambda^{-y}(1)dy = \frac{\Gamma_\lambda(\alpha)}{\Gamma_\lambda(\alpha)} = 1.\end{aligned}$$

Let $X\sim\Gamma_\lambda(\alpha_1,\beta)$ and $Y\sim\Gamma_\lambda(\alpha_2,\beta)$, and let X and Y be independent. Suppose that $U = X+Y,\ V=\frac{XY}{X+Y}$. The joint probability density function of X and Y is given by

$$\begin{aligned}f_{X,Y}(x,y) = f_X(x)f_Y(y) &= \frac{\beta e_\lambda^{-\beta x}(1)}{\Gamma_\lambda(\alpha_1)}\frac{\beta e_\lambda^{-\beta y}(1)}{\Gamma_\lambda(\alpha_2)}(\beta y)^{\alpha_2-1}(\beta x)^{\alpha_1-1} \\ &= \frac{\beta^{\alpha_1+\alpha_2}x^{\alpha_1-1}e_\lambda^{-\beta x}(1)y^{\alpha_2-1}e_\lambda^{-\beta y}(1)}{\Gamma_\lambda(\alpha_1)\Gamma_\lambda(\alpha_2)}.\end{aligned} \tag{12}$$

Let

$$u=g_1(x,y)=x+y, \quad v=g_2(x,y)=\frac{x}{x+y}. \tag{13}$$

Then, from (13), we have

$$\frac{\partial g_1}{\partial x}=1,\ \frac{\partial g_1}{\partial y}=1, \quad \frac{\partial g_2}{\partial x}=\frac{y}{(x+y)^2},\ \frac{\partial g_2}{\partial x_2}=-\frac{x}{(x+y)^2}.$$

From (13), we have $x=uv,\ y=u(1-v)$. Note that

$$\frac{\partial(g_1,g_2)}{\partial(x_1,x_2)}=\det\begin{bmatrix}\frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2}\\ \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x}\end{bmatrix}=\det\begin{bmatrix}1 & 1\\ \frac{y}{(x+y)^2} & \frac{-x}{(x+y)^2}\end{bmatrix}=-\frac{1}{x+y}=-\frac{1}{u}. \tag{14}$$

By (8), (9) and (14), we get

$$\begin{aligned} f_{U,V}(u,v) &= f_{X,Y}(x,y)\left|\frac{\partial(g_1,g_2)}{\partial(x_1,x_2)}\right|^{-1}=uf_{X,Y}(uv,u(1-v)) \\ &= u\frac{e_\lambda^{-\beta uv}(1)}{\Gamma_\lambda(\alpha_1)}\beta^{\alpha_1+\alpha_2}(uv)^{\alpha_1-1}\frac{e_\lambda^{-\beta u(1-v)}(1)}{\Gamma_\lambda(\alpha_2)}\big(u(1-v)\big)^{\alpha_2-1} \\ &= \frac{u^{\alpha_1+\alpha_2-1}\beta\beta^{\alpha_1+\alpha_2-1}}{\Gamma_\lambda(\alpha_1+\alpha_2)}e^{-\beta u}\frac{\Gamma_\lambda(\alpha_1+\alpha_2)}{\Gamma_\lambda(\alpha_1)\Gamma_\lambda(\alpha_2)}v^{\alpha_1-1}(1-v)^{\alpha_2-1} \\ &= \frac{(u\beta)^{\alpha_1+\alpha_2-1}\beta}{\Gamma_\lambda(\alpha_1+\alpha_2)}e^{-\beta u}\frac{\Gamma_\lambda(\alpha_1+\alpha_2)}{\Gamma_\lambda(\alpha_1)\Gamma_\lambda(\alpha_2)}v^{\alpha_1-1}(1-v)^{\alpha_2-1}. \end{aligned} \tag{15}$$

Thus, by (15), we obtain the following theorem which gives the solution to the modified open problem.

Theorem 2.3. *Let $X\sim\Gamma_\lambda(\alpha_1,\beta)$, $Y\sim\Gamma_\lambda(\alpha_2,\beta)$, and let X and Y be independent with the joint probability density function $f_{X,Y}(x,y)$. Suppose that $U=X+Y$, $V=\frac{X}{X+Y}$. Then the joint probability function of U and V is given by*

$$f_{U,V}(u,v)=\frac{(u\beta)^{\alpha_1+\alpha_2-1}\beta}{\Gamma_\lambda(\alpha_1+\alpha_2)}e^{-\beta u}\frac{\Gamma_\lambda(\alpha_1+\alpha_2)}{\Gamma_\lambda(\alpha_1)\Gamma_\lambda(\alpha_2)}v^{\alpha_1-1}(1-v)^{\alpha_2-1}.$$

3. Conclusion

In recent years, the degenerate gamma functions were introduced as a degenerate version of the classical gamma functions and the related degenerate gamma random variables were investigated. In this note, in place of those functions and random variables, we considered the modified degenerate gamma functions and the modified degenerate gamma random variables. Then we solved the 'Modified Open Problem,' as it is stated in Theorem 2.3. This gives a partial answer to the original 'Open Problem,' which is stated in Section 1. The solution was obtained by using the change of variables formula in classical analysis.

Degenerate gamma functions, degenerate umbral calculus, and various degenerate versions of some special numbers and polynomials have been studied in recent years and many interesting results were obtained. As one of our future research projects, we would like to continue to work on degenerate versions of some stuffs and to find their applications to physics, science and engineering as well as to mathematics.

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DEPARTMENT OF MATHEMATICS, KWANGWOON UNIVERSITY, SEOUL 139-701, REPUBLIC OF KOREA
Email address: `tkkim@kw.ac.kr`

DEPARTMENT OF MATHEMATICS, SOGANG UNIVERSITY, SEOUL 121-742, REPUBLIC OF KOREA
Email address: `dskim@sogang.ac.kr`

DEPARTMENT OF APPLIED MATHEMATICS, KYUNG HEE UNIVERSITY, SEOUL 02447, REPUBLIC OF KOREA
Email address: `wjookim@khu.ac.kr`

DEPARTMENT OF MATHEMATICS EDUCATION, GYONGSANG NATIONAL UNIVERSITY, JINJU 52828, REPUBLIC OF KOREA
Email address: `mathkjk26@gnu.ac.kr`